# Appendix to:

# Why Do Demand Curves for Stocks Slope Down?

# A More Elaborate Model

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# 1 A More Elaborate Model

## 1.1 Motivation

Our earlier model provides a simple illustration of our economic story. But its frictionless setup also produces the result that prices in the market are set by long-short investors who take large positions each way and whose net portfolio value can even be negative. Yet in reality we observe relatively small short interest for most stocks.

Dechow et al. (2001) report that about 80% of the firm-years in their sample[1] have a short interest less than 0.5% of shares outstanding, and less than 2% of the firm-years have a short interest greater than 5%. Nowadays short-selling is a little more common, and e.g. for August 15, 2002, the NYSE reported a record short interest of 2.3% of all shares outstanding.[2] Since this figure includes the shares that were shorted for various hedging motives, the average short interest due to fundamental investors (i.e. stock pickers) is even smaller. This general unwillingness to short stocks could arise at least in part as a consequence of the short-sales costs documented by e.g. Jones and Lamont (2002) and D'Avolio (2002).

When a stock is added to the S&P 500 and mechanical indexers buy about 10% of the shares outstanding, most of the supply seems to come from investors who owned the stock before the event. E.g. for the event of July 19, 2002, when seven large U.S. firms replaced seven non-U.S. firms in the index, the average short interest one month before the event, between the announcement and effective days, and one month after the event were 3.0%, 3.2%, and 5.0%, respectively, for the additions, and 2.6%, 2.8%, and 2.2% for the deletions, while the overall NYSE short interest was 2.2%, 2.1%, and 2.3%.[3] While this event suggests that about 2% of the required 10% supply came from short sellers, historically the number is likely to be even smaller.

Hence, most of the fundamental stock valuation and stock-picking clearly has been and still seems to be done by long-only investors rather than unconstrained long-short investors. We can accommodate this by changing the interpretation of our simple model as we do in the extension section of the main paper, or by building it explicitly into the model as we

[1] NYSE and AMEX stocks, 1976-1993.

[2] NYSE press release, available at http://www.nyse.com/.

[3] Data from the exchanges, published monthly by the Wall Street Journal.

do here.

We have three main reasons to build a more elaborate (and realistic) version of the model: First, the new version of the model serves as a robustness check on the results of the simple model. Second, numerical calibration is generally easier and more easily interpretable for a more realistic model setup. Third, it turns out this setup can give us some results even if the end investors are not fully rational.

## 1.2 The Model

The basic setup is the same as before. There is a risk-free asset yielding an interest rate of zero, and $N_S$ stocks with terminal payoffs $\widetilde{x}_i = a_i + b_i\widetilde{y} + \widetilde{e}_i$. The end investors maximize CARA utility by optimally allocating their wealth to active managers, passive managers, and the risk-free asset. Again we abstract entirely from agency issues and let the active managers simply follow the orders they are given.

There are essentially five differences with the simple model presented earlier. First, the active managers can only take long positions in stocks. Second, because of this short-sales constraint, the active managers will be benchmarked against the market portfolio. Third, there are multiple active managers and they have heterogeneous beliefs about stocks so that all stocks will be held in equilibrium. Fourth, we allow for wide dispersion in the operating sizes of firms ($a_i$). Fifth, each active manager will have beliefs about a subset of the stocks but not all of them.

### 1.2.1 Assets

There is enormous dispersion in the market capitalization of firms. If we take only the largest 3,000 stocks (which constitute the Russell 3000 index and still represent less than a half of all stocks listed on the NYSE, AMEX, and Nasdaq) at the end of 2001, we get a distribution of values from about \$130 million to \$400 billion.

We let the constant $a_i$ of the payoff of stock $i$ to be distributed as $\log(a_i) \sim U(\log(a_{\min}), \log(a_{\max}))$. While a lognormal distribution would fit the data better, we pick this form for analytical tractability. What matters is the degree of dispersion, not its exact shape.

The dispersion in $a_i$ almost completely eliminates any size effect from the model. If each $a_i$ had the same value or if their dispersion was very small, then any uninformed investor would be able to earn above-market returns by simply buying the cheaper stocks and shorting the more expensive ones. But when there is large dispersion in the operating size of firms, this simple correlation between market price and expected return is severely diminished, and the uninformed investors will not be able to do better than the market portfolio. The dispersion in $a_i$ effectively ensures that the uninformed investors cannot become informed by just using some piece of easily available information.

The dispersion in $a_i$ also creates dispersion in the dollar supply of idiosyncratic risk. If the same investors know about the same stocks, then the smaller stocks will be more aggressively priced and will have more horizontal demand curves. In reality most of these properties are relatively constant across stocks, so the dispersion in $a_i$ implies that the number of market participants in each stock and their aggregate risk tolerance are also roughly proportional to $a_i$. This is why we cannot allow all investors know about all stocks. This approach is also somewhat similar to Merton (1987).

We assume $b_i = P_i$ and $\sigma_{e_i}^2 = P_i^2 \sigma_i^2$, so that each stock will always have a market beta $\beta_i = 1$ and a fixed return variance of $\sigma_i^2$. These assumptions have a negligible effect on our numerical results but they do make our equations more convenient and intuitive.

### 1.2.2 End Investors

The representative end investor's problem is again

$$\max_{\{W_a, W_p\}} E\left[-\exp\left(-\gamma_e \widetilde{W}_1\right)\right] \tag{1}$$

$$\text{s.t.} \quad \widetilde{W}_1 = W_0 + W_a \widetilde{R}_a + W_p \widetilde{R}_m, \tag{2}$$

which produces the same optimal allocations to the active and passive managers:

$$W_a = \frac{E\left[\widetilde{R}_a\right] - \beta_a \eta}{\gamma_e \sigma_a^2} = \frac{\alpha_a}{\gamma_e \sigma_a^2} \tag{3}$$

$$W_p = \frac{E\left[\widetilde{R}_m\right]}{\gamma_e \sigma_m^2} - \beta_a W_a = \frac{\eta}{\gamma_e \sigma_m^2} - \beta_a W_a. \tag{4}$$

The end investor's allocation to the active managers therefore depends entirely on the alpha $\alpha_a$ (net of fees) of those managers. Whatever market exposure comes from the active portfolio, the end investor fully hedges this by reducing his position in the passive portfolio.

### 1.2.3 Active Managers

There are $K$ active money managers who are all identical ex ante. Therefore the end investor will simply diversify his active portfolio allocation equally across all active managers, giving the manager $k$ an allocation of $W_k = \frac{W_a}{K}$.

The manager $k$ has beliefs about $M$ stocks which are a subset of the $N_S$ stocks available. Specifically, manager $k$'s belief about the payoff $a_i$ of stock $i$ is given by $a_{ik} \sim U(a_i - \Delta a, a_i + \Delta a)$.

For the same reasons as before, we model each manager as a CARA investor with a coefficient of absolute risk aversion $\gamma$. Without loss of generality, we construct $N_S$ uncorrelated hybrid securities with payoffs $\widetilde{z}_i = a_i + \widetilde{e}_i$ and prices $P_{z_i} = P_i(1+\eta)$. This determines the dollar demand of the active manager $k$ for stock $i$:

$$W_{ik} = \max\left\{\frac{1}{\gamma\sigma_i^2}\left[\frac{a_{ik}}{P_i} - (1+\eta)\right], 0\right\}. \tag{5}$$

Hence, his demand is linear in his perceived alpha $\alpha_{ik} = \frac{a_{ik}}{P_i} - (1+\eta)$, or zero if the perceived alpha is negative.

This also reveals why short-sales constraints can only exist in the presence of heterogeneous beliefs. By construction, the average alpha perceived by any investor is zero, so the investor will have a positive demand for about half the stocks and a zero demand for the other half. Thus if all investors have homogeneous information and face short-sales constraints, half the stocks will have zero demand and their prices are not determined in equilibrium.

The manager invests all the wealth $W_k$ under his management in this portfolio, so $W_k = \sum_{i=1}^{M} W_{ik}$ and hence his effective risk aversion is given by

$$\gamma = \frac{1}{W_k}\sum_{i=1}^{M}\frac{\alpha_{ik}}{\sigma_i^2}. \tag{6}$$

Since the end investor is effectively benchmarking the manager against the market portfolio by instructing him to focus on abnormal returns, the manager can ignore the market risk of his portfolio and let the end investors offset this on their own by investing less with the passive managers.

We do not constrain the manager to trade only a subset of $M$ out of the available $N_S$ stocks. However, the average alpha of a stock is zero by construction, so for all the stocks that the manager has no information about, his expected alpha is zero and thus his optimal demand for such stocks is zero. Unlike in Merton (1987), here the incomplete diversification of the active managers results from a restriction on their information sets and not on an explicit restriction on their investment universe. Nevertheless, the exact degree of diversification by the active managers (such as whether they are diversified beyond 50 stocks) does not play a role in any of our results.

Each active manager charges a fee $f$ as a fraction of assets under management.

### 1.2.4 Equilibrium

We define the equilibrium as the set of prices and allocations such that the active managers have invested all their wealth under management in portfolios with mean-variance efficient abnormal returns, the passive managers have invested all their wealth under management in the value-weighted market portfolio, the end investors are maximizing their expected utility by optimally allocating their wealth between the active managers, passive managers, and the risk-free asset, and the market clears for all stocks.

In equilibrium, stock $i$ will be held by the passive managers who hold a supply of $u_p = \frac{W_p}{P_m}$, the noise traders who hold a randomly chosen supply of $u_{in} \sim U(0, \Delta u)$, and the active managers who hold the remaining supply which we denote as $u_i$. Market clearing then requires that

$$u_p + u_{in} + u_i = 1 \tag{7}$$

which implies that $u_i \sim U(u_{\min}, u_{\min} + \Delta u)$ where $u_{\min} = 1 - u_p - \Delta u$.

We assume there is a continuum of managers with a measure of $N_i$ who know about stock $i$. Their total dollar demand for stock $i$ is then

$$W_i = \begin{cases} \int_{a=P_i(1+\eta)}^{a=a_i+\Delta a_i} \frac{1}{\gamma\sigma_i^2}\left[\frac{a}{P_i} - (1+\eta)\right]\frac{N_i}{2\Delta a_i}da & \text{if } P_i \geq \frac{a_i - \Delta a_i}{(1+\eta)} \\ \int_{a=a_i-\Delta a_i}^{a=a_i+\Delta a_i} \frac{1}{\gamma\sigma_i^2}\left[\frac{a}{P_i} - (1+\eta)\right]\frac{N_i}{2\Delta a_i}da & \text{if } P_i < \frac{a_i - \Delta a_i}{(1+\eta)}. \end{cases} \tag{8}$$

In the latter case the price of the stock is below the valuation of even the most pessimistic investor. This is unlikely unless the dispersion in beliefs is very small, so we focus on the latter case where we have both investors who believe the stock has a negative alpha and investors who believe it has a positive alpha.

The price of stock $i$ will then be

$$P_i = \frac{a_i + \Delta a_i}{1 + \eta + 2\sigma_i\sqrt{\gamma\frac{\Delta a_i}{N_i}u_i}} = \frac{a_i(1+\Delta_i)}{1 + \eta + 2\sigma_i\sqrt{\frac{\Delta_i\gamma}{\lambda_i}u_i}}, \tag{9}$$

where we defined the relative dispersion-of-beliefs parameter $\Delta_i = \frac{\Delta a_i}{a_i}$ and the density of informed investors $\lambda_i = \frac{N_i}{a_i}$. This determines the true alpha (i.e., conditional on $a_i$) of stock $i$ as

$$\alpha_i = \frac{1}{1+\Delta_i}\left[2\sigma_i\sqrt{\frac{\Delta_i\gamma}{\lambda_i}u_i} - \Delta_i\left(1+\eta\right)\right]. \tag{10}$$

Analogously to the results of e.g. Miller (1977) and Chen, Hong, and Stein (2002), the price of the stock reflects the valuation $a_i\left(1+\Delta_i\right)$ of the most optimistic investor. However, this valuation is discounted by $\eta + 2\sigma_i\sqrt{\frac{\Delta_i\gamma}{\lambda_i}u_i}$ which is greater than the market risk premium $\eta$ and which reflects the active investors' aversion to idiosyncratic risk, so that the average alpha across all stocks is still equal to zero.

The above equations determine the joint distribution of stock prices and alphas as a function of the minimum fraction $u_{\min}$ of a stock held by the active managers, the effective risk aversion $\gamma$ of the active managers, and the market risk premium $\eta$, in addition to some stock-specific constants. They also have to be consistent with the equilibrium allocations of $W_a$ and $W_p$ to the active and passive managers. These five variables have to be solved for simultaneously from the following system of five equations:

$$\alpha_m = 0 \tag{11}$$

$$W_a = \frac{\alpha_a}{\gamma_e\sigma_a^2} \tag{12}$$

$$W_p = \frac{\eta}{\gamma_e\sigma_m^2} - W_a \tag{13}$$

$$\gamma = \frac{K}{W_a}\sum_{i=1}^{M}\frac{\alpha_{ik}}{\sigma_i^2} \tag{14}$$

$$u_{\min} = 1 - \frac{W_p}{P_m} - \Delta u \tag{15}$$

Here $\alpha_m$ denotes the alpha of the value-weighted market portfolio and $P_m$ is the price of the market portfolio.

To solve this system of equations, we first need to compute several expressions: the average alpha $\alpha_m$ of the market portfolio, the average alpha $\alpha_a$ (net of fees) of the active managers, the idiosyncratic variance $\sigma_a^2$ of the active managers, and the summation $\sum_{i=1}^{M}\frac{\alpha_{ik}}{\sigma_i^2}$ for an active manager. These computations do not lend themselves to easy and intuitive economic interpretation. Hence, we solve for the equilibrium numerically.

Intuitively, the equilibrium is established as follows: Assume we start in an equilibrium with some fee $f$ which determines the equilibrium allocations $W_a$ and $W_p$ and the equilibrium distributions of stock prices and alphas. Then suddenly the fee is increased to $f'$. Now the active managers can no longer earn their fees, so the end investors will reduce their dollar allocation to the active managers. Once the dollar allocation of the active managers decreases, they become less aggressive, permitting a wider equilibrium distribution of alphas (in equation (10), decreasing the active managers' equilibrium holding $u_i$ while keeping its variation unchanged will increase the dispersion of alphas). This wider distribution of alphas will increase the average alpha of the active managers. Once the average alpha rises to the same level as the new fee $f'$, a new equilibrium is reached.

Thus, the intuition of our simple model generalizes to the richer and more realistic model. Here the mechanics of the model are more complicated, but in return we get parameter values and predictions that are easier to interpret (the share of wealth controlled by active managers; no unrealistically high values of short interest).

### 1.3 Analysis of Equilibrium

As before, we calibrate the model by setting the number of stocks $N_S = 1,000$, the risk aversion of the end investors $\gamma_e = 1.5625 \times 10^{-5}$ (to produce a market risk premium of $\eta = 0.05$), and the dispersion in noise trader holdings $\Delta u = 0.4$. We also set $\beta_i = 1$ and the standard deviation $\sigma_i = 0.3$ for the idiosyncratic return for all stocks, and the standard deviation $\sigma_m = 0.2$ for the market return.

We pick $a_{\min} = 1$ and $a_{\max} = 688$ so that the average $a_i$ is still equal to 105 (as before) but now there is large dispersion around this mean value. We set the mass of active managers $K = 10$ and we let each active manager know about $M = 100$ stocks. Then the average measure of managers who know about stock $i$ is $N_i = \frac{KM}{N_S} = 1$, and we assume this is proportional to the expected payoff $a_i$ which implies a density $\lambda_i = \frac{1}{105}$ of active managers for all stocks. The scaling of the number of managers is of course irrelevant as we do the calculations for a continuum of managers. Finally, we choose the maximum dispersion of beliefs $\Delta_i$ for a stock as 20% of the expected payoff $a_i$.

The meaningful free parameters to be picked in the model are the active managers' fee $f$, the dispersion of beliefs $\Delta_i$, and the dispersion of noise traders' demand $\Delta u$. The model's restrictions then determine the joint distributions of $u_i$ (the supply held by active managers), $P_i$, and $\alpha_i$, as well as the allocations $W_a$ and $W_p$ to the active and passive managers, the active managers' effective risk aversion $\gamma$, and most importantly the slope of the demand curve. The calibration results are in Table 1.

| fee | $\frac{W_a}{W_a+W_p}$ | $[\alpha_{min}, \alpha_{max}]$ | effective risk aversion $\gamma$ | price impact of a $-10\%$ supply shock |
|---|---|---|---|---|
| 0.01% | 425% | $[-0.52\%, 0.51\%]$ | $1.72 \times 10^{-3}$ | 0.25% |
| 0.1% | 135% | $[-1.7\%, 1.6\%]$ | $5.42 \times 10^{-3}$ | 0.80% |
| 0.5% | 61% | $[-3.9\%, 3.5\%]$ | $1.21 \times 10^{-2}$ | 1.8% |
| 1.0% | 44% | $[-5.8\%, 4.9\%]$ | $1.72 \times 10^{-2}$ | 2.7% |
| 1.5% | 36% | $[-7.4\%, 5.9\%]$ | $2.12 \times 10^{-2}$ | 3.3% |
| 2.0% | 32% | $[-8.9\%, 6.8\%]$ | $2.46 \times 10^{-2}$ | 3.9% |

Table 1: The effect of the management fee; one-year horizon.

For a realistic cost of 1.5% of assets under management, the end investors would allocate 36% of their stock market wealth to professional stock pickers and 64% to passive strategies. The price impact following a $-10\%$ supply shock would be 3.3%, or about 3 times as large as in our simple model. Compared with the CAPM benchmark, the order-of-magnitude difference is still due to the same story as before, i.e. the fact that the costly delegation of portfolio management severs the link between the market risk premium and cross-sectional stock pricing. However, the short-sales constraints in this model give a further nontrivial boost to the slope of the demand curve, although this clearly does not change its order of magnitude.

When the fee of the active managers tends to zero, the price impact does seem to approach zero and the demand curves become close to horizontal. This also shows up as a very aggressive allocation to the active managers. Convergence in this model is complicated by the fact that a very small fee and consequently a very large allocation to the active managers (financed by shorting the passive managers) leads to the active managers' portfolio becoming more and more like the market portfolio. Hence, the idiosyncratic variance of the portfolio falls at the same time as the alpha of the portfolio falls, partially offsetting the effect from a lower average alpha. So while the model does approach the simple CAPM case with almost horizontal demand curves as the fee tends to zero, the model produces more interesting predictions for more realistic values of the fee.

The slope of the demand curve will be steeper if we decrease the dispersion in noise trader holdings $\Delta u$ or increase the dispersion in beliefs or active managers $\Delta_i$. Since the differences in pricing in the cross-section are distributed over the interval of noise trader holdings $[0, \Delta u]$, a narrower interval will mean that the demand curve will have to be steeper to produce the same equilibrium dispersion in alphas. The dispersion in beliefs $\Delta_i$ enters through the breadth-of-ownership intuition of Chen, Hong, and Stein (2002): As the supply available to the active managers decreases towards zero, only the valuation of the most optimistic manager determines the stock price since the others cannot short the stock. As the supply available to the managers then increases from zero and the price starts to fall, a wide dispersion in beliefs means it takes a greater fall in price to induce the same number of managers to jump in and hold a positive position in the stock. Nevertheless, the model is relatively robust to changes in these two parameters.

As before, increasing the horizon from one to five years will roughly multiply the price impacts by five. Thus the magnitude of the actual index premium is not outside the scope of this model.

Even if the end investors are not fully rational, we can still use this model to describe the slope of the demand curve, given some (not perfectly rational) allocations to the active and passive managers. E.g. if the end investors allocate a little over a third of their wealth to professional stock pickers and invest the rest in the market portfolio or in random portfolios, we would get similar results as in the equilibrium with rational end investors and a fee of 1.5%. Demand curves would still slope down because of the delegation of portfolio management, i.e. because the active managers are constrained to invest no more than 100% of their wealth under management and because the end investors determine the market risk premium separately from the cross-sectional pricing. However, the puzzle about the demand curves then becomes a puzzle about why the end investors do not invest more with active managers who earn positive alphas. The introduction of the fee for active management can provide a rational explanation for this asset allocation puzzle.

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